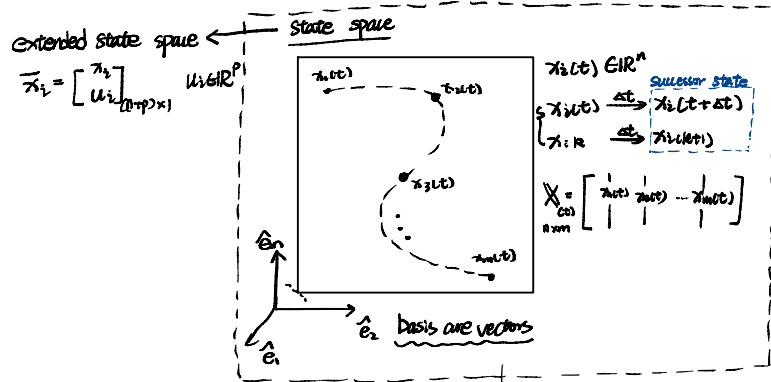


Koopman theory - Part 1: state space & lifted (observable) space



transition mapping

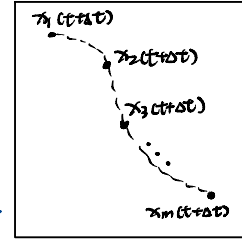
$x_i(t+\Delta t) = F_{\Delta t}(x_i(t))$

$x_i(k+\Delta t) = F_{\Delta t}(x_i(k))$

flow map

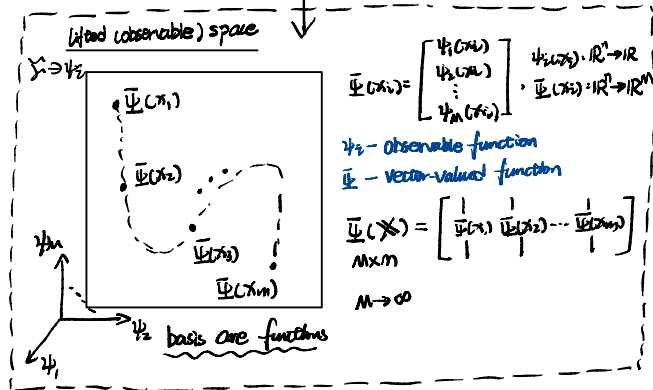
$F_{\Delta t}$: evolution operator

nonlinear



ψ_i

$\bar{\Psi}$: all square-integrable real-valued functions



$\bar{\Psi}(x_i(t+\Delta t)) = K_{\Delta t} \bar{\Psi}(x_i(t))$

$\bar{\Psi}(x_i(k+\Delta t)) = K_{\Delta t} \bar{\Psi}(x_i(k))$

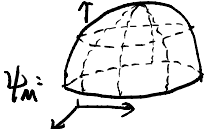
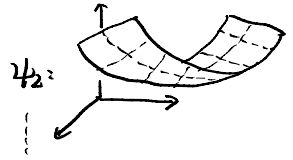
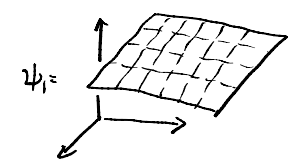
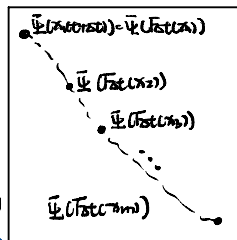
$\bar{\Psi}(x_i(k+\Delta t)) = \bar{\Psi}(F_{\Delta t}(x_i(k)))$

$K_{\Delta t}$: discrete-Koopman operator

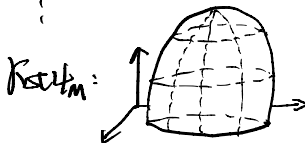
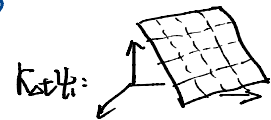
operate on function

$K_{\Delta t} \bar{\Psi} = \bar{\Psi} \circ F_{\Delta t}$

linear



$K_{\Delta t} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{bmatrix} = \begin{bmatrix} K_{\Delta t} \psi_1 \\ K_{\Delta t} \psi_2 \\ \vdots \\ K_{\Delta t} \psi_m \end{bmatrix}$



g_0 : scalar-valued observable function

g : vector-valued observable function

represented in observable space

① observables as basis $\bar{\Psi} \in \bar{\psi}_i^M$

$g_0(x) = a_1 \psi_1(x) + a_2 \psi_2(x) + \dots + a_m \psi_m(x) = \sum_{i=1}^m a_i \psi_i(x)$

$= [a_1, a_2, \dots, a_m] \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_m(x) \end{bmatrix} = \bar{a}^T \cdot \bar{\Psi}(x) = \bar{a}^T \bar{\Psi}(x)$

$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i} \psi_i(x) \\ \sum_{i=1}^m a_{2i} \psi_i(x) \\ \vdots \\ \sum_{i=1}^m a_{ni} \psi_i(x) \end{bmatrix} = \sum_{i=1}^m \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \psi_i(x) = \sum_{i=1}^m \bar{a}_{ci} \psi_i(x)$

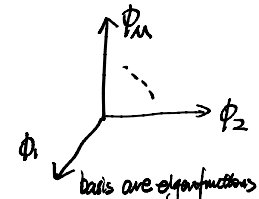
where $a_{ji} = \langle g_j, \psi_i \rangle$

$= \begin{bmatrix} \bar{a}_{c1} \cdot \bar{\Psi}(x) \\ \bar{a}_{c2} \cdot \bar{\Psi}(x) \\ \vdots \\ \bar{a}_{cn} \cdot \bar{\Psi}(x) \end{bmatrix} = \begin{bmatrix} \bar{a}_{c1} \\ \bar{a}_{c2} \\ \vdots \\ \bar{a}_{cn} \end{bmatrix} \bar{\Psi}(x) = A \cdot \bar{\Psi}(x)$

where $A = [\bar{a}_{c1} \dots \bar{a}_{cn}]$

$\Rightarrow g(x_k) = (K_{\Delta t})^k g(x_0) = (K_{\Delta t})^k \sum_{i=1}^m \bar{a}_{ci} \psi_i(x_0) = (K_{\Delta t})^k \cdot A \cdot \bar{\Psi}(x_0)$

② eigenfunctions as basis $\bar{\Phi} \in \phi_i^M$, $K_{\Delta t} \bar{\Phi} = \lambda \bar{\Phi}$, $K_{\Delta t} \phi_i = \lambda_i \phi_i$



set $v \rightarrow a$ $\phi_i \rightarrow \psi_i$

$g(x) = \bar{v}^T \bar{\Phi}(x) = \bar{v}^T \bar{\Psi}(x)$

$g(x) = \sum_{i=1}^m \bar{v}_{ci} \phi_i(x) = \begin{bmatrix} \bar{v}_{c1} \\ \vdots \\ \bar{v}_{cm} \end{bmatrix} \bar{\Phi}(x) = V \cdot \bar{\Phi}(x)$

Koopman mode

Koopman theory - Part I: Koopman operator & eigenfunction, eigenvalue, eigenmode

dynamic system equation: $\frac{dx}{dt} = f(x, u, t)$ - nonlinear

discrete - Koopman operator: $K_{\Delta t}$ $K_{\Delta t} \Psi(x) = \Psi(F_{\Delta t}(x))$ $K_{\Delta t} \Phi(x) = M \Phi(x)$

$\{\mu_i, \phi_i, \vec{v}_i\}$ $\phi_i(x)$ - eigenfunction, μ_i - eigenvalue

$$\Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}, M = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_n \end{bmatrix}$$

continuous - Koopman operator: $\mathcal{L}$

$\{\lambda_i, \phi_i, \vec{v}_i\}$

$$\mathcal{L} \Psi(x) = \lim_{\Delta t \rightarrow 0} \frac{K_{\Delta t} \Psi(x) - \Psi(x)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Psi(F_{\Delta t}(x)) - \Psi(x)}{\Delta t}$$

$$= \frac{\partial \Psi(x)}{\partial t} = \frac{\partial \Psi(x)}{\partial x} \frac{\partial x}{\partial t} = \nabla_x \Psi(x) \cdot f$$

① relationship between $K_{\Delta t}$ & $\mathcal{L}$: $K_{\Delta t} = e^{\Delta t \mathcal{L}}$

$$\mathcal{L} := \frac{\partial}{\partial t} \quad \frac{\partial \Psi}{\partial t} = \mathcal{L} \cdot \Psi$$

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial \Psi}{\partial t} \right) = \frac{\partial}{\partial t} (\mathcal{L} \Psi) = \mathcal{L} (\mathcal{L} \Psi) = \mathcal{L}^2 \Psi$$

$$\vdots$$

$$\frac{\partial^n \Psi}{\partial t^n} = \mathcal{L}^n \Psi$$

$$K_{\Delta t} \Psi(x) = \Psi(F_{\Delta t}(x)) = \Psi(x(t + \Delta t)) = \Psi(t) + \frac{\partial \Psi(x(t))}{\partial t} \frac{\Delta t}{1} + \frac{\partial^2 \Psi(x(t))}{\partial t^2} \frac{\Delta t^2}{2!} + \dots$$

$$= \Psi(x(t)) + \mathcal{L} \Psi(x(t)) \frac{\Delta t}{1} + \mathcal{L}^2 \Psi(x(t)) \frac{\Delta t^2}{2!} + \dots$$

$$= (1 + \mathcal{L} \frac{\Delta t}{1} + \mathcal{L}^2 \frac{\Delta t^2}{2} + \dots) \cdot \Psi(x(t))$$

$$= e^{\Delta t \mathcal{L}} \Psi(x(t))$$

② relationship of eigenfunction & eigenvalue: $\mu_i = e^{\Delta t \lambda_i} \cdot \phi_i$

$$\mathcal{L} \phi_i = \lim_{\Delta t \rightarrow 0} \frac{K_{\Delta t} \phi_i - \phi_i}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mu_i \phi_i - \phi_i}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(e^{\Delta t \lambda_i} - 1) \phi_i}{\Delta t} = \lambda_i \phi_i$$

$$\therefore \mathcal{L} \Phi(x) = \frac{\partial \Phi(x)}{\partial t} = \Lambda \Phi(x), \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

③ properties:

$$K_{\Delta t} \cdot \phi_i(x_k) = \phi_i(x_{k+1}) = \mu_i \phi_i(x_k)$$

$$K_{\Delta t} \cdot (\alpha_1 \Phi_1(x) + \alpha_2 \Phi_2(x)) = \alpha_1 \Phi_1(F_{\Delta t}(x)) + \alpha_2 \Phi_2(F_{\Delta t}(x))$$

$$K_{\Delta t} (\phi_i(x) \phi_j(x)) = \phi_i(F_{\Delta t}(x)) \phi_j(F_{\Delta t}(x)) = \mu_i \mu_j \phi_i(x) \phi_j(x) = e^{\Delta t \lambda_i} e^{\Delta t \lambda_j} \phi_i(x) \phi_j(x)$$

$$= e^{\Delta t (\lambda_i + \lambda_j)} \phi_i(x) \phi_j(x)$$

$$\mathcal{L} (\phi_i(x) \phi_j(x)) = \frac{\partial \phi_i(x) \phi_j(x)}{\partial t} = \frac{\partial \phi_i(x)}{\partial t} \cdot \phi_j(x) + \phi_i(x) \frac{\partial \phi_j(x)}{\partial t}$$

$$= \lambda_i \phi_i(x) \phi_j(x) + \phi_i(x) \cdot \lambda_j \phi_j(x) = \lambda_i \phi_i(x) \phi_j(x) + \lambda_j \phi_i(x) \phi_j(x)$$

$$= (\lambda_i + \lambda_j) \phi_i(x) \phi_j(x)$$

Krylov space

$$g(x_k) = (K_{\Delta t})^k g(x_0) = (K_{\Delta t})^k \cdot \sum \vec{v}_i \cdot \phi_i(x) = (K_{\Delta t})^k \cdot V \cdot \Psi(x)$$

$$= (K_{\Delta t})^k \cdot \sum \vec{v}_i \cdot \Psi(x) = (K_{\Delta t})^k \cdot A \cdot \Psi(x)$$

$$g(x_{k+1}) = K_{\Delta t} g(x_k) = (K_{\Delta t} g)(x_k) = \sum \vec{v}_i \cdot (K_{\Delta t} \phi_i)(x_k) = \sum \mu_i \vec{v}_i \cdot \phi_i(x_k)$$

$\parallel$
 $e^{\Delta t \lambda_i}$

$\Phi(k+1)$

Koopman theory - Part III: approximate Koopman-operator (finite-dimensional) by EDMD & Koopman mode

$$g_0(x) = \vec{a}_c \cdot \vec{\Psi}(x) \quad \text{scalar - one observable function}$$

$$K_{\text{st}} \cdot g_0(x) = K_{\text{st}} (\vec{a}_c \cdot \vec{\Psi}(x)) = \vec{a}_c \cdot (K_{\text{st}} \vec{\Psi}(x)) \approx \vec{a}_c \cdot [K_{\text{st}} \vec{\Psi}(x) + r(x)]$$

$\downarrow$
 approximate Koopman operator

① K_{st} : matrix $M \times M$ $K_{\text{st}} \cdot \vec{\Psi}(x) \leftrightarrow (K_{\text{st}} \vec{\Psi})(x)$: EDMD $\Upsilon = [\vec{y}_1 \dots \vec{y}_M]$

Dictionary dataset Snapshots $\{(y_i, z_i)\}$

$$y_i = \psi_i(x(t)) \in \mathbb{R}^M$$

$$z_i = \psi_i(x(t+\Delta t)) \in \mathbb{R}^M$$

$$Z = [\vec{z}_1 \dots \vec{z}_M]$$

$$K_{\text{st}} \cdot g_0(x) = \vec{a}_c \cdot (K_{\text{st}} \vec{\Psi}(x) + r(x))$$

$$J = \frac{1}{2} \sum_{i=1}^M |r(x_i)|^2 = \frac{1}{2} \sum_{i=1}^M \left| \vec{a}_c \cdot (K_{\text{st}} \vec{\Psi}(x_i) - \vec{\Psi}(x_{i+1})) \right|^2 \leftarrow \text{least-square question}$$

$$\Rightarrow \|K_{\text{st}} - Z \cdot \Upsilon^\dagger\| = \|\vec{\Psi}(x_{\text{test}}) \cdot \vec{\Psi}(x_{\text{test}})^\dagger\|$$

or $\sum_i (z_i \cdot y_i^*)(y_i \cdot y_i^*)^\dagger$

② $K_{\text{st}} \cdot \varepsilon_i = \mu_i \varepsilon_i$ $K_{\text{st}} \cdot \vec{\Theta} = \vec{\Theta} M$, $\vec{\Theta} = [\vec{\varepsilon}_1 \dots \vec{\varepsilon}_M]$ $M = [\mu_1 \dots \mu_M]$

$$W_i^* \cdot K_{\text{st}} = \mu_i W_i^* \quad W_i^* \cdot K_{\text{st}} = M W_i^* , W_i^* = \begin{bmatrix} -W_i^* \\ -W_i^* \end{bmatrix}$$

• $W_i^* K_{\text{st}} = M W_i^* \Rightarrow K_{\text{st}} = (W_i^*)^\dagger M W_i^* \Rightarrow K_{\text{st}} (W_i^*)^\dagger = (W_i^*)^\dagger M$; $(W_i^*)^\dagger = \vec{\Theta}$

$$\therefore W_i^* \cdot \vec{\Theta} = \begin{bmatrix} -W_i^* \\ -W_i^* \end{bmatrix} \cdot [\vec{\varepsilon}_1 \dots \vec{\varepsilon}_M] = I \Rightarrow \boxed{W_i^* \cdot \varepsilon_i = 1}$$

• $K_{\text{st}} \cdot \phi_i(x) = \mu_i \phi_i(x) \approx \vec{b}_c \cdot K_{\text{st}} \vec{\Psi}(x) \leftarrow \text{assumption from EDMD}$

$$\mu_i \vec{b}_c \cdot \vec{\Psi}(x)$$

$$\therefore \mu_i \vec{b}_c \cdot \vec{\Psi}(x) = \vec{b}_c \cdot K_{\text{st}} \vec{\Psi}(x) \Rightarrow \mu_i \vec{b}_c = \vec{b}_c \cdot K_{\text{st}} \Rightarrow \vec{b}_c = W_i^*$$

$$\Rightarrow \mu_i W_i^* \cdot \vec{\Psi}(x) = W_i^* \cdot K_{\text{st}} \vec{\Psi}(x) = \mu_i \phi_i(x) \Rightarrow \boxed{W_i^* \cdot \vec{\Psi}(x) = \phi_i(x)} \quad \phi_i(x) = z_i \vec{\Psi}(x)$$

$$\Rightarrow W_i^* \cdot \vec{\Psi}(x) = \phi_i(x) , \quad \langle \vec{\Psi}, \phi_i \rangle : \text{Koopman mode}$$

$$\vec{\Psi}(x) = \vec{\Theta} \vec{\Phi}(x) = \begin{bmatrix} \vec{\varepsilon}_1 \dots \vec{\varepsilon}_M \end{bmatrix} \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_M(x) \end{bmatrix} = \sum_{i=1}^M \langle \vec{\varepsilon}_i, \vec{\Psi}(x) \rangle \phi_i(x)$$

• $K_{\text{st}} \cdot \vec{\Psi}(x) = K_{\text{st}} \cdot \sum_{i=1}^M \varepsilon_i \phi_i(x) = \sum_{i=1}^M \mu_i \varepsilon_i \phi_i(x)$

$$K_{\text{st}} \phi_i(x) = K_{\text{st}} \cdot W_i^* \cdot \vec{\Psi}(x) = W_i^* (K_{\text{st}} \vec{\Psi}(x)) = \mu_i \phi_i(x)$$

③ $g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_M(x) \end{bmatrix} = V \cdot \vec{\Psi}(x) = V \cdot W^* \cdot \vec{\Psi}(x) \Rightarrow g(x) = \sum_{i=1}^M \vec{v}_{ci} \cdot \phi_i(x)$

$$\uparrow \langle g, \vec{v}_{ci} \rangle$$

$$= A \cdot \vec{\Psi}(x) = A \cdot \vec{\Theta} \vec{\Phi}(x) \Rightarrow g(x) = \sum_{i=1}^M \vec{a}_{ci} \cdot \phi_i(x)$$

$$\uparrow \langle g, \vec{a}_{ci} \rangle$$

$$\therefore V = A \vec{\Theta} , A = V \cdot W^*$$

$$\therefore [\vec{v}_{c1} \dots \vec{v}_{cM}] = [\vec{a}_{c1} \dots \vec{a}_{cM}] \cdot \vec{\Theta} : \vec{v}_{ci} = \vec{a}_{ci} \cdot \vec{\Theta} : i\text{th Koopman mode}$$

$$K_{\text{st}} \cdot g(x) = K_{\text{st}} V \cdot \vec{\Psi}(x) = \sum_{i=1}^M K_{\text{st}} \cdot \phi_i(x) \vec{v}_{ci} = \sum_{i=1}^M \mu_i \phi_i(x) \vec{v}_{ci}$$

$$K_{\text{st}} \cdot g(x) = K_{\text{st}} \cdot A \cdot \vec{\Psi}(x) = \sum_{i=1}^M K_{\text{st}} \cdot \psi_i(x) \vec{a}_{ci} = \sum_{i=1}^M z_i \vec{a}_{ci}$$

④ Koopman mode:

$$\vec{\Psi}(x) = \vec{H} \vec{\Phi}(x)$$

$g(x)$: any general function:

$$g(x) = A \cdot \vec{\Psi}(x)$$

$$g(x) = V \cdot \vec{\Phi}(x) \Rightarrow V W^* = A$$

full-state observable function: $g(x) =$

DMD: dynamic mode decomposition (Tu 2014, Brunton's book)

① standard DMD: schmid 2008

data sets: $\{x_0, \dots, x_m\} \quad x_i \in \mathbb{R}^n$, in sequential time series: $z_i \xrightarrow{\Delta t} z_{i+1}$

$$a) Y = \begin{bmatrix} x_0 & \dots & x_{m-1} \end{bmatrix}, Z = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}$$

$$b) Y = \tilde{U} \tilde{\Sigma} \tilde{V}^T \quad (r \text{ reduced}) \quad A\tilde{\Phi} = \tilde{\Phi} \Lambda, \tilde{\Phi} = \text{dynamic mode}$$

$$c) Z = A \cdot Y, \quad A = Z Y^T = Z \tilde{V} \tilde{\Sigma}^+ \tilde{U}^* \quad (* \text{ conjugate transpose})$$

d) PCA mode: $\tilde{x}_k = \tilde{U}^* x_k$: project the data into reduced subspace (PCA mode)
capture the most common features (projected DMD mode)

$$x_{k+1} = A x_k \quad \begin{cases} \tilde{x}_k = \tilde{U}^* x_k \Rightarrow \tilde{U} \tilde{x}_{k+1} = A \tilde{U} \tilde{x}_k \Rightarrow \tilde{x}_{k+1} = \tilde{U}^* A \tilde{U} \tilde{x}_k \\ \tilde{x}_{k+1} = \tilde{U}^* x_{k+1} \end{cases}$$

$$\tilde{x}_{k+1} = \tilde{A} \tilde{x}_k, \quad \tilde{A} = \tilde{U}^* A \tilde{U} = \tilde{U}^* Z \tilde{V} \tilde{\Sigma}^+$$

$$e) \tilde{A} \tilde{\Xi} = \tilde{\Xi} \Lambda$$

f) calculate the DMD mode $\Phi, A\Phi = \Phi \Lambda$

i) $\Phi = \tilde{U} \tilde{\Xi}$ (schmid's) — projected DMD mode

$$A\Phi = \tilde{U} \tilde{A} \tilde{U}^* \cdot \tilde{U} \tilde{\Xi} = \tilde{U} \tilde{\Xi} \Lambda = \Phi \Lambda$$

Brunton's book P278: $\tilde{\Xi} = \tilde{U} \tilde{\Xi}$ are not exact eigenvectors

$$\Phi_1 = \tilde{U} \tilde{\Xi} \rightarrow \text{column space of } Y$$

$$\Phi_2 = Z \tilde{V} \tilde{\Sigma}^+ \tilde{W} \rightarrow \text{column space of } Z \quad \checkmark$$

$$A\Phi = \Phi \Lambda \rightarrow \Phi \text{'s column space} = A \text{'s column space} \quad ; A = Z Y^T$$

$$\Phi \text{'s column space} = Z \text{'s column space} \quad \checkmark$$

for standard DMD: $Y = [x_0 \dots x_{m-1}] \quad Z = [x_1 \dots x_m] \leftarrow \text{only two columns are different}$

for exact DMD: $Y = [x_0(t) \dots x_m(t)] \quad Z = [x_1(t+\Delta t) \dots x_m(t+\Delta t)] \leftarrow \text{all columns are different}$

$\Rightarrow$ choose $\tilde{\Xi}_1$ is more natural

• when $\mu_i = 0$, if exact DMD mode $\Phi_i = 0$, then we use projected DMD mode.

• DMD mode corresponding to zero eigenvalues are usually not of interest, since they do not play a role in the dynamic

$$x_{k+1} = A \cdot x_k = A^{k+1} \cdot x_0, \quad A = \Phi \Lambda \Phi^T \rightarrow A^{k+1} = \Phi \Lambda^{k+1} \Phi^T \Rightarrow x_{k+1} = \Phi \Lambda^{k+1} \tilde{\Phi}^T x_0, \text{ set } \tilde{b} = \tilde{\Phi}^T x_0$$

$$x_{k+1} = \Phi \Lambda^{k+1} \tilde{b} = \sum \phi_i \mu_i^{k+1} b_i = \sum e^{i \omega_i t} \cdot b_i \cdot \phi_i = \Phi e^{\Lambda t} \tilde{b}, \text{ where } x_0 = \Phi \tilde{b}$$

$$\tilde{U} x_0 = Z \tilde{V} \tilde{\Sigma}^+ \tilde{b} \rightarrow x_0 = \tilde{U}^* Z \tilde{V} \tilde{\Sigma}^+ \tilde{b} \Rightarrow x_0 = \tilde{A} \tilde{\Xi} \tilde{b} = \tilde{\Xi} \Lambda \tilde{b} \Rightarrow \tilde{b} = (\tilde{\Xi} \Lambda)^+ x_0$$

g) $x_{k+1} = \Phi \Lambda^{k+1} \tilde{b}$ set continuous eigenvalues, $x_m = A x_k$

$$\frac{dx(t)}{dt} = \frac{x_{k+1} - x_k}{\Delta t} = \frac{A x_k - x_k}{\Delta t} = \frac{A - I}{\Delta t} \cdot x_k = B \cdot x_k, \quad B = \frac{A - I}{\Delta t}$$

$$\Rightarrow x(t) = e^{Bt} x(0); \quad e^{B \Delta t} = A \Rightarrow B = \frac{\log(A)}{\Delta t} \Rightarrow \lambda = \frac{\log(\mu)}{\Delta t}$$

② exact DMD: Tu's or Brunton's

$\rightarrow$ change the data set: a) $Y = [x_0(t) \dots x_m(t)] \quad Z = [x_1(t+\Delta t) \dots x_m(t+\Delta t)]$
no need to be sequential time series

d): $\tilde{x}_k(t) = \tilde{U}^* x_k(t) \quad \tilde{x}_k(t+\Delta t) = \tilde{U}^* x_k(t+\Delta t) \Rightarrow$ when $x_k(t+\Delta t) = x_{k+1}(t)$, then standard DMD is a special case of exact DMD

$$f) \quad i) \Phi = Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi} \quad \text{— exact DMD mode}$$

$$A\Phi = Z \tilde{V} \tilde{\Sigma}^+ \tilde{U}^* \cdot Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi} = Z \tilde{V} \tilde{\Sigma}^+ \tilde{\tilde{A}} \tilde{\Xi} = Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi} \Lambda = \Phi \Lambda$$

③ Exact DMD, alternative method $\rightarrow$ understand how we use projection

$$[Y \quad Z] = Q R, \quad Q^T Q = I \Rightarrow \text{more costly}$$

$$d) \text{ new projected mode: } \begin{cases} \tilde{x}_k(t+\Delta t) = Q^T x_k(t+\Delta t) \\ \tilde{x}_k(t) = Q^T x_k(t) \end{cases} \Rightarrow x_k(t+\Delta t) = A x_k(t)$$

$$\Rightarrow \tilde{A} = Q^T A Q \quad Q \cdot \tilde{x}_k(t+\Delta t) = A Q \cdot \tilde{x}_k(t)$$

$$\tilde{x}_k(t+\Delta t) = \tilde{Q}^T \tilde{A} Q \tilde{x}_k(t)$$

$$\Rightarrow \tilde{A} = Q^T A Q$$

$$e) \tilde{A} \tilde{\Xi} = \tilde{\Xi} \Lambda$$

$$f) \Phi = Q \tilde{\Xi}, \quad A\Phi = Q Q^T A \Phi = Q Q^T A Q \tilde{\Xi} = Q \tilde{\tilde{A}} \tilde{\Xi} = Q \tilde{\Xi} \Lambda = \Phi \Lambda$$

④ Exact DMD, sequential time series: $\{x_k\}$

set $Q = [\tilde{U} \quad \tilde{V}]$ where $\tilde{V} = \frac{P}{\|P\|}$ $P = x_m - \tilde{U} \tilde{U}^* x_m \leftarrow$ project x_m on $\{x_0, \dots, x_m\}$

$$f) \Phi = \tilde{U} \tilde{\Xi} \Lambda + \tilde{V} \tilde{V}^* (Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi})$$

if $P=0$, x_m belongs to the column space of $\tilde{U} \Rightarrow Q = \tilde{U}$, exact DMD is identical to projected DMD

$$\Phi = Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi} = Q Q^T (Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi}) = \tilde{U} \tilde{U}^* (Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi}) + \tilde{V} \tilde{V}^* (Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi}) = \tilde{U} \tilde{\Xi} \Lambda + \tilde{V} \tilde{V}^* (Z \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi})$$

$\tilde{\Phi}$: exact DMD mode

DMDc: Dynamic mode decomposition with control — (Brunton, 2014)

$$\text{set } Y = [\tilde{u}_1 \dots \tilde{u}_m] \quad \tilde{u}_i \in \mathbb{R}^p \quad Y = [x_1(t) \dots x_m(t)] \quad Z = [x_1(t+\Delta t) \dots x_m(t+\Delta t)]$$

$$\text{the process dynamic matrix the input matrix}$$

$$Z = A Y + B U \leftarrow x_k(t+\Delta t) = A x_k(t) + B u_k \quad (\text{linear dynamical system})$$

① map B is known find dynamic modes and eigenvalues of underlying system A

$$a) Y = \tilde{U} \tilde{\Sigma} \tilde{V}^*, \quad Z - B U = A Y$$

$$b) A = (Z - B U) Y^T = (Z - B U) \tilde{V} \tilde{\Sigma}^+ \tilde{U}^* \xrightarrow[\text{Y=0}]{\text{when } \mu_i=0} \text{equivalent to DMD}$$

$$c) x_k(t+\Delta t) = A x_k + B u_k$$

$$d) \tilde{U} x_k = \tilde{x}_k \Rightarrow \tilde{U} x_k(t+\Delta t) = A \tilde{U} x_k + B u_k$$

$$\Rightarrow \tilde{x}_k(t+\Delta t) = \tilde{U}^* A \tilde{U} \tilde{x}_k + \tilde{U}^* B u_k = \tilde{A} \tilde{x}_k + \tilde{B} u_k$$

$$\Rightarrow \begin{cases} \tilde{A} = \tilde{U}^* A \tilde{U} = \tilde{U}^* (Z - B U) \tilde{V} \tilde{\Sigma}^+ \\ \tilde{B} = \tilde{U}^* B \end{cases}$$

$$e) \tilde{A} \tilde{\Xi} = \tilde{\Xi} \Lambda$$

$$f) \tilde{\Phi} = (Z - B U) \tilde{V} \tilde{\Sigma}^+ \tilde{\Xi}, \quad A \tilde{\Phi} = \tilde{\Phi} \Lambda \leftarrow \text{same as DMD}$$

② map B is unknown: approximation of the matrix A and B can both be found from state and control snapshots

$$a) Z = A Y + B U \Rightarrow Z = [A \quad B] \begin{bmatrix} Y \\ U \end{bmatrix} = G \cdot \Omega$$

$$b) \Omega = \tilde{U}_n \tilde{\Sigma}_n \tilde{V}_n^* \quad (\text{truncated by } n)$$

$$c) G * Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* = [A \quad B] = [Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^*, Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^*]$$

$$d) Z = \tilde{U}_z \tilde{\Sigma}_z \tilde{V}_z^* \quad (\text{truncated by } r) \leftarrow r > n$$

Unlike DMD, the truncated left singular vectors $\tilde{U}_n$ cannot be used to define the subspace on which the state evolves. (the truncated left singular vectors of Ω define the input space).

To find a linear transformation P for the states $x \rightarrow P x$, we utilise the reduced-order subspace of the output subspace

$$e) x_k = \tilde{U}_z x_k \Rightarrow \tilde{U}_z x_k(t+\Delta t) = A \tilde{U}_z x_k(t) + B u_k$$

$$\tilde{x}_k(t+\Delta t) = \tilde{U}_z^* A \tilde{U}_z \tilde{x}_k(t) + \tilde{U}_z^* B u_k$$

$$\tilde{x}_k(t+\Delta t) = \tilde{A} \tilde{x}_k(t) + \tilde{B} u_k$$

$$\Rightarrow \tilde{A} = \tilde{U}_z^* A \tilde{U}_z = \tilde{U}_z^* Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \tilde{U}_z$$

$$\tilde{B} = \tilde{U}_z^* B = \tilde{U}_z^* Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \tilde{U}_z$$

$$f) \tilde{A} \tilde{\Xi} = \tilde{\Xi} \Lambda$$

$$g) \tilde{\Phi} = Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \tilde{U}_z \tilde{\Xi}, \quad A \tilde{\Phi} = \tilde{\Phi} \Lambda$$

$$A\tilde{\Phi} = Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \cdot (\tilde{U}_z \tilde{U}_z^*) \cdot Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \tilde{\Xi}$$

$$= Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \cdot \tilde{U}_z \tilde{A} \tilde{\Xi}$$

$$= Z \tilde{V}_n \tilde{\Sigma}_n^+ \tilde{U}_n^* \cdot \tilde{U}_z \tilde{\Xi} \Lambda$$

$$= \tilde{\Phi} \Lambda$$

EDMDc (Koopman with inputs and control — KIC)

$$x_k \xrightarrow{\Phi} \tilde{\Phi}_k(x_k) \xrightarrow{m} \tilde{\Phi}_k(x_k) \quad \begin{bmatrix} \tilde{\Phi}_k(x_k) \\ \tilde{\Phi}_k(u_k) \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_k(x_k) \\ \tilde{\Phi}_k(u_k) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_k(x_k) \\ \tilde{\Phi}_k(u_k) \end{bmatrix}$$

delay coordinate

$$y_i = \begin{bmatrix} \text{out}(t) \\ \text{out}(t-\Delta t) \\ \text{out}(t-2\Delta t) \end{bmatrix} \quad Y = \begin{bmatrix} y_1 & \dots & y_m \end{bmatrix} \quad K_e = Z \cdot Y^T$$

$$z_i = \begin{bmatrix} \text{in}(t) \\ \text{in}(t-\Delta t) \\ \text{in}(t-2\Delta t) \end{bmatrix} \quad Z = \begin{bmatrix} z_1 & \dots & z_m \end{bmatrix}$$

full state

$$y_k = \begin{bmatrix} x_k(t) \\ u_k(t) \end{bmatrix} \quad z_k = \begin{bmatrix} x_k(t+\Delta t) \\ u_k(t+\Delta t) \end{bmatrix}$$

linear dynamic system with full state observable

$$x_{i(t+1)} = A \cdot x_{i(t)} + B u_{i(t)} \iff x_{i(t+1)} = A \cdot x_{i(t)} + B u_{i(t)}$$

$$\begin{bmatrix} x_{i(t+1)} \\ u_{i(t+1)} \end{bmatrix} = \underline{y}(x_{i(t+1)}, u_{i(t+1)}) = K_{\text{est}} \cdot \underline{y}(x_{i(t)}, u_{i(t)}) = \begin{bmatrix} \hat{K}_{\text{est}11} & \hat{K}_{\text{est}12} \\ \hat{K}_{\text{est}21} & \hat{K}_{\text{est}22} \end{bmatrix} \cdot \begin{bmatrix} x_{i(t)} \\ u_{i(t)} \end{bmatrix}$$

$$\Rightarrow K_{\text{est}} = \underset{K_{\text{est}}}{\text{argmin}} \left(\left\| \overrightarrow{\underline{y}(x_{i(t+1)}, u_{i(t+1)})} - \overrightarrow{K_{\text{est}}^* \cdot \underline{y}(x_{i(t)}, u_{i(t)})} \right\|_2^2 + \lambda \cdot \left\| \overrightarrow{K_{\text{est}}} \right\|_1 \right) = \text{LASSO}$$